Python and Maths Basics

Exercise 1 : Machine Learning (general)

(a) Define the terms "supervised learning", "unsupervised learning", and "reinforcement learning".

Answer

- Supervised: Learn a function from a set of input-output pairs
- Unsupervised: Identify structures in unlabeled data.
- Reinforcement: Maximizing rewards

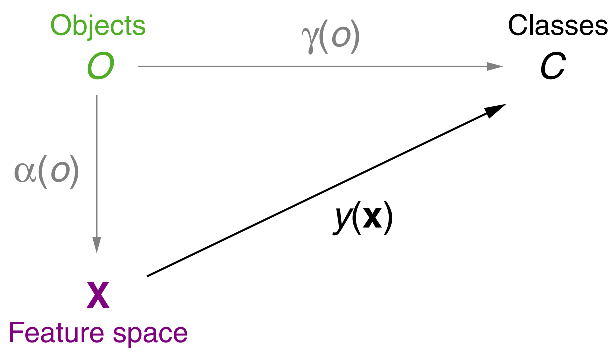
(b) Sketch for each learning paradigm a typical problem together with a description of its technical realization.

Answer

- Supervised learning. Text classification, Spam filtering. - given a database of emails, each annotated as "spam" or "not spam", learn a function that classifies new emails
- Unsupervised learning. Sort fruits in a basket, clear up a writing-desk, compression of data.
- Reinforcement learning. Replace a light bulb, touch a hot hot plate.

Exercise 2 : Specification of Learning Tasks

The following picture from the lecture slides describes the relationship between Real World and Model World, when it comes to the specification of learning tasks.



(a) Assume you are building a machine learning system that predicts whether a given mushroom is poisonous or edible. For the following list, decide which symbol from the picture most closely matches the given list item:

(a) A pile of Mushrooms.

(b) A table with the columns "size", "weight", and "color", as well as one row for each possible mushroom, and the respective measurements in the cells.

(c) A human mushroom expert who can tell whether any mushroom you show them is poisonous or edible.

(d) A device that measures size, weight and color of a mushroom.

(e) The set $\{\mathsf{Poisonous}, \mathsf{Edible}\}$

(f) The machine learning system that you are trying to build.

Answer

(a) O, (b) $\mathbf{X}$, (c) $\gamma(o)$, (d) $\alpha(o)$, (e) C, (f) $y(\mathbf{x})$.

Exercise 3 : Linear Algebra

The three $(2,3)$-matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are given as

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 2 \\ 1 & 4 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 8 & -1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & -1 & 2 \\ 5 & -2 & 0 \end{bmatrix}.$$

Evaluate the following expressions:

(a) $3\mathbf{A} + 2\mathbf{B} - 5\mathbf{C}$

Answer

$$\begin{aligned} 3\mathbf{A} + 2\mathbf{B} - 5\mathbf{C} &= 3\begin{bmatrix} 3 & 5 & 2 \\ 1 & 4 & 0 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 & 2 \\ 5 & 8 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & -1 & 2 \\ 5 & -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 15 & 6 \\ 3 & 12 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 4 \\ 10 & 16 & -2 \end{bmatrix} + \begin{bmatrix} -5 & 5 & -10 \\ -25 & 10 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 22 & 0 \\ -12 & 38 & -2 \end{bmatrix} \end{aligned}$$

(b) $3(\mathbf{B} - \mathbf{A})^T - \mathbf{C}^T$

Answer

$$\begin{aligned} 3(\mathbf{B} - \mathbf{A})^T - \mathbf{C}^T &= 3\left(\begin{bmatrix} 2 & 1 & 2 \\ 5 & 8 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 5 & 2 \\ 1 & 4 & 0 \end{bmatrix}\right)^T - \begin{bmatrix} 1 & -1 & 2 \\ 5 & -2 & 0 \end{bmatrix}^T \\ &= 3\begin{bmatrix} -1 & -4 & 0 \\ 4 & 4 & -1 \end{bmatrix}^T - \begin{bmatrix} 1 & -1 & 2 \\ 5 & -2 & 0 \end{bmatrix}^T \\ &= 3\begin{bmatrix} -1 & 4 \\ -4 & 4 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 5 \\ -1 & -2 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 12 \\ -12 & 12 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 1 & 2 \\ -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 7 \\ -11 & 14 \\ -2 & -3 \end{bmatrix} \end{aligned}$$

(c) $\mathbf{A} \cdot \mathbf{C}^T$

Answer

$$\mathbf{A} \cdot \mathbf{C}^T = \begin{bmatrix} 3 & 5 & 2 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 5 & -2 & 0 \end{bmatrix}^T$$
$$= \begin{bmatrix} 3 & 5 & 2 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & -2 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 5 \\ -3 & -3 \end{bmatrix}$$

Exercise 4 : Calculus

Calculate the first-order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of the following functions:

(a) $z = (2x - 3y^2)^5$

Answer

$$z = u^5 \text{ with } u = 2x - 3y^2$$
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = 5u^4 \cdot 2 = 10u^4 = 10(2x - 3y^2)^4$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = 5u^4 \cdot (-6y) = -30yu^4 = -30y(2x - 3y^2)^4$$

(b) $z = x^2 \cdot e^{-xy}$

Answer

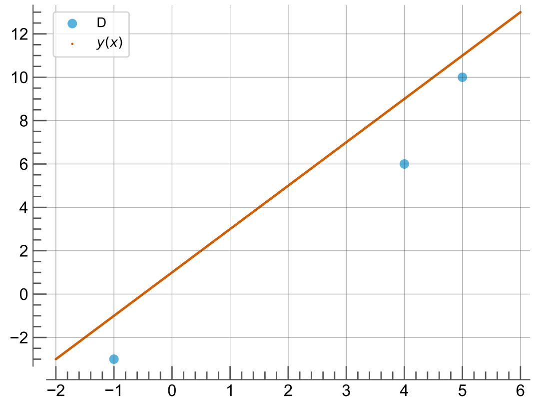
$$z = uv \text{ with } u = x^2, v = e^{-xy}$$
$$\frac{\partial u}{\partial x} = 2x$$
$$v = e^t \text{ with } t = -xy$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial t} \cdot \frac{\partial t}{\partial x} = e^t \cdot (-y) = -y \cdot e^{-xy}$$
$$\frac{\partial z}{\partial x} = u_x v + v_x u = 2x \cdot e^{-xy} - y \cdot e^{-xy} \cdot x^2 = (2x - x^2 y) \cdot e^{-xy}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial y} = x^2 \cdot e^t \cdot (-x) = -x^3 \cdot e^{-xy}$$

Exercise 5 : Gradient Descent

In this exercise you will be calculating one iteration of the LMS algorithm. The task is to fit a straight line given as $y(x) = w_0 + w_1 x$ through a given set of points D by adjusting the parameters w_0 and w_1. The pairs $(\mathbf{x}, c) \in D$ are given as $(4, 6), (-1, -3), (5, 10)$.

(a) Plot the line parametrized with $w_0 := 1, w_1 := 2$ as well as all samples from D into the same coordinate system.

Answer



(b) For the first iteration, $(4, 6)$ is 'randomly' selected as the ground-truth pair $(\mathbf{x}, c)$. Compute the loss w.r.t. $\mathbf{w}$, which is defined as

$$l_2(c, y(\mathbf{x})) = \frac{1}{2} \cdot (c - y(\mathbf{x}))^2.$$

Answer

$$\begin{aligned} l_2(c, y(\mathbf{x})) &= \frac{1}{2} \cdot (c - y(\mathbf{x}))^2 \\ &= \frac{1}{2} \cdot (c - (w_0 + w_1 \cdot x))^2 \\ &= \frac{1}{2} \cdot (6 - (w_0 + w_1 \cdot 4))^2 \\ &= \frac{1}{2} \cdot (6 - w_0 - 4w_1)^2 \end{aligned}$$

(c) Derive the gradient

$$\begin{pmatrix} \frac{\partial l_2}{\partial w_0} \\ \frac{\partial l_2}{\partial w_1} \end{pmatrix}.$$

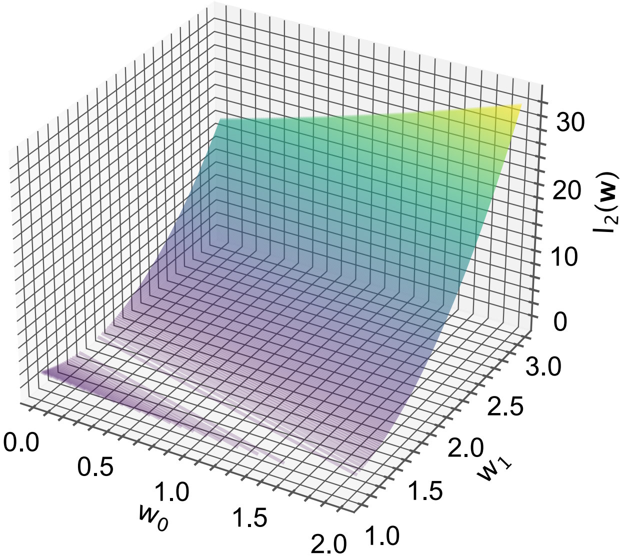
Verify that this is indeed $-\delta \cdot \mathbf{x}$.

Answer

$$\begin{aligned} \frac{\partial l_2}{\partial w_0} &= \frac{1}{2} \cdot 2 \cdot (6 - w_0 - 4w_1) \cdot (-1) \\ &= -6 + w_0 + 4w_1 \\ \frac{\partial l_2}{\partial w_1} &= \frac{1}{2} \cdot 2 \cdot (6 - w_0 - 4w_1) \cdot (-4) \\ &= -24 + 4w_0 + 16w_1 \end{aligned}$$

$$\begin{aligned} \delta &= c - y(\mathbf{x}) = 6 - (w_0 + w_1 \cdot 4) = 6 - w_0 - 4w_1 \\ -\delta \cdot \mathbf{x} &= \begin{pmatrix} -(6 - w_0 - 4w_1) \cdot x_0 \\ -(6 - w_0 - 4w_1) \cdot x_1 \end{pmatrix} \\ &= \begin{pmatrix} -(6 - w_0 - 4w_1) \cdot 1 \\ -(6 - w_0 - 4w_1) \cdot 4 \end{pmatrix} \\ &= \begin{pmatrix} -6 + w_0 + 4w_1 \\ -24 + 4w_0 + 16w_1 \end{pmatrix} \end{aligned}$$

(d) The following plot shows the loss landscape defined by l_2 with the current choice of $(\mathbf{x}, c) = (4, 6)$.



(d1) The model is initialized with $w_0 := 1$, $w_1 := 2$. Mark the current location of the model in the loss landscape, i.e., $\begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$.

(d2) Draw the line of gradient descent, which is defined as

$$\underbrace{\left\{ \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} - \eta \cdot \begin{pmatrix} \frac{\partial l_2}{\partial w_0} \\ \frac{\partial l_2}{\partial w_1} \end{pmatrix} \mid 0 \leq \eta \leq 0.03 \right\}}_{\mathbf{w}+\boldsymbol{\Delta}\mathbf{w} \text{ for increasing } \eta},$$

by connecting the start and end point from the set.

Answer